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Quantum Logics with Given Centers and Variable State Spaces

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We ask which logics with a given center allow for enlargements with an arbitrary state space. We show that these are precisely those logics the center of which possesses a two-valued state and the state space of which is nonempty. This extends the results of Binder as well as our previous results and supplements the results of Foulis and Pták and of Navara, Pták, and Rogalewicz. We also comment on some related questions.

1. INTRODUCTION AND BASIC NOTIONS

In the logicoalgebraic approach to the foundation of quantum mechanics we usually identify the "quantum logic" of an experiment with a σ-orthocomplete orthomodular poset and the "state" of an experiment with a σ-additive probability measure (e.g., Gudder, 1979; Pták and Pulmannová, 1991; Varadarajan, 1968). The set of all "absolutely compatible" events of an experiment is then identified with the center of the corresponding σ-orthocomplete orthomodular poset. A natural question arises whether this identification allows for independence of the state space and the center (it should be noted that, for instance, in the von Neumann algebra formalism this is not the case; see Kadison, 1965). We want to show in this paper that it is "almost" so. Under a very mild condition on the center we are able to guarantee a full independence. Moreover, we also ensure an arbitrary degree of noncompatibility in our constructions (i.e., in proving that there is a quantum logic with a given center and an arbitrary state space, we ensure that such a logic may be as "nonclassical" as needed).

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The constructions are rather technical in places. Routine details are therefore sometimes omitted. Nevertheless, all essential ideas are at least sketched. The reader interested exclusively in orthomodular lattices can simply replace the word "logic" with "orthomodular lattice"—the results remain valid without changes.

Let us recall some basic notions as we shall use them in the sequel. By a *quantum logic* (or simply by a *logic*) we mean a partially ordered set $L = (L, \leq)$ with a least element 0 and a greatest element 1 and with a complementation operation $'$ such that the following conditions are satisfied:

(i) $a = a''$, and if $a \leq b$, then $b' \leq a'$ $(a, b \in L)$.
(ii) If $a_i \in L$ $(i \in N)$ and $a_i \leq a_j'$ for any $i,j \in N$ $(i \neq j)$, then $\vee_{i \in N} a_i$ exists in L.
(iii) If $a \leq b$, then $a \vee (b \wedge a') = b$ $(a, b \in L)$.

Let us reserve the symbol L for logics. Typical examples of logics are Boolean σ-algebras or lattices of projections in a von Neumann algebra. We do not assume that quantum logics are necessarily lattices.

Let us define the notion of *sublogic* of a logic. Let K and L be logics. Then K is said to be a sublogic of L if there is an injective mapping $e\colon K \to L$ such that:

(i) $e(0) = 0$.
(ii) $e(a') = e(a)'$ $(a \in K)$.
(iii) $e(\vee_{i \in N} a_i) = \vee_{i \in N} e(a_i)$ ($a_i \in K$ for any $i \in N$) provided $a_i \leq a_j'$ $(i \neq j)$.
(iv) $a \leftrightarrow b$ (in K) if and only if $e(a) \leftrightarrow e(b)$ (in L) [recall that the symbol $a \leftrightarrow b$ means that a is *compatible* with b, i.e., $a = (a \wedge b) \vee (a \wedge b')$ and $b = (a \wedge b) \vee (b \wedge a')$].

If K is a sublogic of L, then K is a logic in its own right with the operations inherited from L. Thus, in this case we can understand K as a subset of L. If K is a Boolean σ-algebra, we call it a Boolean sublogic of L. The *center* of L, $C(L)$, is defined to be the intersection of all maximal Boolean sublogics of L. Of course, $C(L)$ is again a Boolean sublogic of L. Moreover, $a \in C(L)$ if and only if $a \leftrightarrow b$ for any $b \in L$.

Let L be a logic. By *a state* on L we mean a probability measure on L. Thus, a mapping $s\colon L \to [0, 1]$ is a state on L if:

(i) $s(1) = 1$.
(ii) $s(\vee_{i \in N} a_i) = \Sigma_{i \in N} s(a_i)$ whenever $a_i \in L$ $(i \in N)$ and $a_i \leq a_j'$ $(i \neq j)$.

Let us denote by $\mathcal{S}(L)$ [resp. $\mathcal{S}_2(L)$] the set of all states (resp. the set of all two-valued states) on L. The set $\mathcal{S}(L)$ naturally carries an affine and a topological structure inherited from the space $[0, 1]^L$. The set $\mathcal{S}(L)$ endowed with this affine and topological structure is called the *state space* of L. It is easily seen that $\mathcal{S}(L)$ is convex [obviously, $\mathcal{S}(L)$ is almost never compact].

There is a characterization of state spaces of logics among convex sets. It was proved in Navara and Rüttimann (1991) that *the state spaces are exactly s-semiexposed faces* in compact convex subsets of locally convex Hausdorff topological spaces (or, equivalently, in a product of real lines). [Let us only recall the definition of an s-semiexposed face. Consider the affine hull $A(K)$ of a compact convex set K and its second dual $A(K)^{**}$. An element $f \in A(K)^{**}$ is said to be an s-functional if f is the weak* limit of an isotone sequence of elements in $[0, e] \cap A(K)$ (e is the unit function on K). A face F of K is said to be s-exposed if there exists an s-functional f such that $F = f^{-1}(1) \cap K$, and F is said to be s-semiexposed if F is an intersection of s-exposed faces of K.]

We are ready to state our result. It is convenient to employ the following notion (naturally, the sign "=" means the existence of an affine homeomorphism when applied for state spaces, the existence of a Boolean σ-isomorphism when applied for Boolean σ-algebras).

Definition. Let L be a logic and let B be a Boolean σ-algebra. Then L is called *state-space-flexible with B fixed for the center* if the following condition is satisfied: If we are given an s-semiexposed face F, then there is a logic K such that:

(i) L is a sublogic of K.
(ii) $C(K) = B$.
(iii) $\mathcal{S}(K) = F$.

Theorem. Let L be a logic and let B be a Boolean σ-algebra. Then L is state-space-flexible with B fixed for the center if and only if $\mathcal{S}(L) \neq \emptyset$ and $\mathcal{S}_2(B) \neq \emptyset$.

Proof. Let us first show that the condition is necessary. Obviously, $\mathcal{S}(L) \neq \emptyset$, because if it is not the case, then L cannot be embedded into any logic which possesses states. Let us show that $\mathcal{S}_2(B) \neq \emptyset$. We have to verify that if L is state-space-flexible with B fixed for the center, then there is a two-valued state on B. Take a singleton for F. Thus, set $F = \{0\}$. Obviously, F is an s-semiexposed face in the interval $[0, 1]$. Suppose that K is such a logic that allows for an embedding $e: L \to K$ and, moreover, suppose that $C(K) = B$ and $\mathcal{S}(K) = F$. If $c \in C(K)$, then evidently $K = [0, c]_K \times [0, c']_K$. (Here the symbols $[0, c]_K$, $[0, c']_K$ denote the corresponding intervals in K.) Consider now the state spaces $\mathcal{S}([0, c]_K)$ and $\mathcal{S}([0, c']_K)$. If $\mathcal{S}([0, c]_K) \neq \emptyset$ and also $\mathcal{S}([0, c']_K) \neq \emptyset$, then we can easily construct two different states on K. This is absurd. Thus, one of these state spaces must be empty. Let us say that $\mathcal{S}([0, c]_K) = \emptyset$. Then $\mathcal{S}([0, c']_K)$ is a singleton. This means that for the restriction $\bar{s}$ of the state s, $s \in \mathcal{S}(K)$, to $C(K)$, we have $\bar{s}(c) = 0$ and $\bar{s}(c') = 1$. This can be applied to every element of $C(K)$, obtaining a two-valued state on $C(K)$. Since $C(K) = B$, we have $\mathcal{S}_2(B) \neq \emptyset$, which we wanted to show.

Let us show that the condition is sufficient. The proof follows to certain extent the technique of Binder (1986) and Navara and Pták (1988); we therefore omit technical details. Let us divide the proof into a few parts.

A. Suppose that F is a singleton and $B = \{0, 1\}$. Then we claim that L can be embedded into a logic K such that $C(K) = \{0, 1\}$ and $\mathcal{S}(K) = F$. In order to obtain the proof of **A**, let us make the following series of constructions.

(a) Fix a state on L, some $s \in \mathcal{S}(L)$. Choose an $\mathcal{M}$-base M in L. [Let us recall that an $\mathcal{M}$-base is such a collection M, $M \subset L$, that the following conditions are satisfied: (i) $a \in M$, $a \leq b \Rightarrow b \in M$, (ii) $a \notin M \Rightarrow a' \in M$. An $\mathcal{M}$-base exists in every logic and each $\mathcal{M}$-base contains no orthogonal pair (Katrnoška, 1982; Marlow, 1978). We will now enlarge the logic L in such a way that all states on the larger logic will be uniquely determined by their values attained on elements of L. Since the values of a state on L are already determined by the values on M, we will obtain an enlargement with a singleton state space.

(b) Suppose that $r \in [0, 1]$. Then there is a logic L_r such that $\mathcal{S}(L_r)$ is a singleton and, moreover, there is an atom $a_r \in L_r$ such that *the only* state $s_r \in \mathcal{S}(L_r)$ satisfies the condition $s_r(a_r) = r$. This construction is demonstrated, e.g., in Pták (1987).

(c) Suppose that $a \in M'$, where M is the chosen $\mathcal{M}$-base and $M' = \{x | x = y' \text{ for some } y \in M\}$. We will now use the following construction maneuver which will indicate the way to enlarge the logic L and also will make the value of any state of the enlargement equal to $s(a)$ [recall that $s \in \mathcal{S}(L)$ is the fixed state]. Put $r = s(a)$ and construct the logic L_r (containing an atom a_r) with the properties listed in part (b). We will utilize the construction called in Navara and Rogalewicz (1991) a replacement of an atom a_r in L_r with a logic $I_a = [0, a]_L$. This construction proves the following proposition. There is a logic L^a with the following properties:

(i) There is an isomorphism h_a of L_r and a sublogic of L^a.
(ii) There is an isomorphism i_a of I_a and $[0, h_a(a_r)]_{L^a}$.
(iii) The logic L^a is generated by $h_a(L_r) \cup i_a(I_a)$.

The construction of such a logic is presented in detail in Navara and Rogalewicz (1991). [All elements of the set-theoretic difference $L^a - h_a(L_r)$ can be expressed as orthogonal suprema $h_a(b) \vee i_a(c)$, where $b \in [0, a_r']_{L_r}$, $c \in I_a$.]

Finally, we identify the corresponding elements of the isomorphic sublogics $[0, a]_L \cup [a', 1]_L$ and $[0, h(a_r)]_{L^a} \cup [h(a_r)', 1]_{L^a}$. Each state on L^a is uniquely determined by its values on $[0, h(a_r)]_{L^a}$ ($= [0, a]_L$) and attains the value r on a_r ($= a$).

(d) Suppose that we have constructed the logic L^a for any $a \in M'$. Thus, we now have a family

$$\mathcal{P} = \{L\} \cup \{L^a | \, a \in M'\}$$

We will perform the pasting of the whole family $\mathcal{P}$, reaching the logic K we need. This can be done [as verified in Navara and Rogalewicz (1991) and Navara and Pták (1988)]. Let us only indicate the basic ideas of the construction. Observe first that the following assertions hold true for all $R, S \in \mathcal{P}$, $R \neq S$:

(i) If $R, S \in \mathcal{P}$, then $R \cap S$ is a sublogic of both R and S, so that the orderings (resp. the orthocomplementations) of R and S coincide on $R \cap S$.
(ii) If $x \in R \cap S$, then either $[0, x]_R = [0, x]_S \subset R \cap S$ or $[x, 1]_R = [x, 1]_S \subset R \cap S$ (this follows from the fact that M is an $\mathcal{M}$-base).
(iii) $R \cap S$ is closed under the formation of countable orthogonal suprema.
(iv) $R \cap S \subset L$.

Put $K = \cup \mathcal{P}$ and define the partial ordering and orthocomplementation so that $a \leq b$ in K (resp., $a = b'$ in K) if and only if $a \leq b$ (resp., $a = b'$) in some $R \in \mathcal{P}$. It is not difficult to verify that K endowed with this partial ordering and orthocomplementation is a logic. Obviously, $\mathcal{S}(K)$ is a singleton [$\mathcal{S}(K)$ contains exactly the (unique) extension of s over K!]. This completes the proof of Proposition **A.**

B. Thus, for the given logic L we have constructed a logic K such that L is a sublogic of K and the state space $\mathcal{S}(K)$ is a singleton. Suppose now that we are given an arbitrary s-semiexposed face F. By Navara and Rüttimann (1991), there is a non-Boolean logic T such that $\mathcal{S}(T) = F$. Take the horizontal sum of T and K. Denote this logic by V. Then, again, $\mathcal{S}(V) = F$ and $C(V) = \{0, 1\}$. Moreover, L is a sublogic of V. We will now show that there is a logic Q such that $\mathcal{S}(Q) = F$, $C(Q) = B$, and such that V is a sublogic of Q. This will complete the proof.

Take first a finite stateless logic Z (Greechie, 1971). Thus, $\mathcal{S}(Z) = \emptyset$ and Z is finite. Let us denote by Y the horizontal sum of V and Z. Then $\mathcal{S}(Y) = \emptyset$ and V is a sublogic of Y.

Consider now the Boolean σ-algebra B. By the Loomis–Sikorski theorem (Sikorski, 1969) there is Boolean σ-algebra Σ of subsets of a set S so that

B is a σ-epimorphic image of Σ. In other words, there is a Boolean σ-epimorphism $h: \Sigma \to B$. By our assumption, there is a two-valued state on B. Let us denote it by t. Then the composition mapping $h \circ t: \Sigma \to \{0, 1\}$ is a two-valued state on Σ. Obviously, the set $\mathcal{F} = \{A \in \Sigma | \ t(h(A)) = 1\}$ forms an ultrafilter on Σ which has the countable intersection property. We will use this ultrafilter in the construction that follows.

Fix a state u on V. Let us consider the set of functions f: S $\to Y$ which have the following properties:

(i) There is an $F_f \in \mathcal{F}$ such that $f|F_f$ is a constant function attaining a value in V.

(ii) For all $z \in Y - V = Z - \{0, 1\}$, the set $U_z = f^{-1}(z)$ belongs to $\Sigma - \mathcal{F}$.

(iii) The composition function $f \circ u$: $(S - \cup_{z \in Z} U_z) \to [0, 1]$ is measurable with respect to Borel subsets of [0, 1] and the σ-algebra $\tilde{\Sigma}$, where $\tilde{\Sigma}$ is the trace of Σ on the set $S - \cup_{z \in Z} U_z$.

It is not difficult to check that the set of all functions f: S $\to Y$ with the above properties (i)–(iii) is a logic (the most technical part is the verification of σ-orthocompleteness, which follows from the fact that measurable functions are closed under the formation of countable sums). Let us denote this logic by W. Since $f \in C(W)$ if and only if $f(S) \subset \{0, 1\}$, we see that $C(W) = \Sigma$. Moreover, $\mathcal{S}(W) = \mathcal{S}(V)$, since Z is stateless and therefore, for any state $w \in \mathcal{S}(W)$, $w(f)$ depends only on the value of f on a set F_f from the ultrafilter $\mathcal{F}$. Since on F_f the function f attains a constant value and this value belongs to V, we have $\mathcal{S}(W) = \mathcal{S}(V)$.

Let us finally factorize the logic W with respect to the σ-ideal of all functions from W which attain nonzero values only on the sets which belong to the kernel $h^{-1}(0)$ of the Loomis–Sikorski σ-epimorphism $h: \Sigma \to B$. We obtain a logic Q such that $C(Q) = B$. Moreover, $\mathcal{S}(Q) = \mathcal{S}(V)$, since no set belonging to the ultrafilter $\mathcal{F}$ can be in the kernel of h. The proof is complete.

By a slight modification of our construction, we can prove the following result of separate, purely algebraic interest.

Proposition. Let L be a logic and let B be a Boolean σ-algebra. Let $\mathcal{S}(L) \neq \emptyset$. Then L can be embedded in a logic K such that $C(K) = B$. (The result remains valid if we write "σ-orthocomplete orthomodular lattice" instead of "logic.")

We conjecture that the above result is valid for all logics L and all Boolean σ-algebras, i.e., we conjecture that the assumption $\mathcal{S}(L) \neq \emptyset$ is not necessary [observe that in the finite additive setup of the problem it is so—the bounded Boolean power does the job; see, e.g., Bruns *et al.* (1990) and Pták

(1987)]. Let us formulate the lattice version (resp. the complete lattice version) of this conjecture as an open question.

Open Question. Suppose that L is a σ-orthocomplete orthomodular lattice and suppose that B is a Boolean σ-algebra. Is there a σ-orthocomplete orthomodular lattice K such that L is a sub-σ-orthocomplete orthomodular lattice of K and $C(K) = B$? Is there a positive answer to this question if we replace "σ-orthocomplete" with "orthocomplete"?

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REFERENCES

Binder, J. (1986). On the interplay of the centre and the state space in quantum logics, *Reports on Mathematical Physics,* **24**, 337–341.

Bruns, G., Greechie, R. J., Harding, J., and Roddy, M. (1990). Completions of orthomodular lattices, *Order,* **7**, 67–76.

Foulis, D., and Pták, P. (1995). On absolutely compatible elements and hidden variables in quantum logics, *Ricerche di Matematica,* **44**, 19–29.

Greechie, R. J. (1971). Orthomodular lattices admitting no states, *Journal of Combinatorial Theory (A),* **10**, 119–132.

Gudder, S. (1979). *Stochastic Methods in Quantum Mechanics*, North-Holland, Amsterdam.

Kadison, R. V. (1965). Transformations of states in operator theory and dynamics, *Topology,* **3** (Supplement 2), 177–198.

Katrnoška, F. (1982). On the representation of orthocomplete posets, *Commentationes Mathematicae Universitatis Carolinae,* **23**, 489–498.

Marlow, A. R. (1978). Quantum theory and Hilbert space, *Journal of Mathematical Physics,* **19**, 1–15.

Navara, M., and Pták, P. (1988). Enlargements of logics (σ-orthocomplete case), In *Proceedings Conference Topology and Measure*, V, Greifswald, Germany, pp. 109–115.

Navara, M., and Rogalewicz, V. (1991). The pasting constructions for orthomodular posets, *Mathematische Nachrichten,* **154**, 157–168.

Navara, M., and Rüttimann, G. (1991). A characterization of σ-state spaces of orthomodular lattices, *Expositiones Mathematicae,* **9**, 275–284.

Navara, M., Pták, P., and Rogalewicz, V. (1988). Enlargements of quantum logics, *Pacific Journal of Mathematics,* **135**, 361–369.

Pták, P. (1987). Exotic logics, *Colloquium Mathematicum,* **54**, 1–7.

Pták, P., and Pulmannová, S. (1991). *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht.

Sikorski, R. (1969). *Boolean Algebras*, Springer-Verlag, Berlin.

Varadarajan, V. S. (1968). *Geometry of Quantum Theory I*, Van Nostrand, Princeton, New Jersey.